Quantum Information Theory

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qubit TV [Gastonjuarez, Wikimedia]

Exercise 3.1: Quantum mechanics: Basics (4P)

Consider the operator

$$\mathbf{A} = \begin{pmatrix} a & b+ic \\ b-ic & a \end{pmatrix} \qquad (a,b,c \in \mathbb{R})$$

(a) Determine the eigenvalues $\lambda_{1,2}$ and the normalized eigenvectors $|\psi_{1,2}\rangle$. (1P)

(b) Perform a spectral decomposition of $\mathbf{A}$, i.e., express $\mathbf{A}$ as a linear combination of the projection operators onto the eigenvectors. (2P)

(c) Suppose that $\mathbf{A}^2 = \mathbf{A}$. What does it imply for a, b, c and the eigenvalues $\lambda_{1,2}$? (1P)

Exercise 3.2: Randomly chosen measurement devices (6P)

Consider a qubit with the orthogonal basis states $|0\rangle$ and $|1\rangle$ and let $|\pm\rangle = \frac{1}{\sqrt{2}}\big(|0\rangle \pm |1\rangle\big)$. Let us consider two measurement devices $\mathbf{A} = |0\rangle\langle 0| - |1\rangle\langle 1|$ and $\mathbf{B} = |+\rangle\langle +| - |-\rangle\langle -|$. In this exercise we investigate a random measurement M which works as follows: We first choose either $\mathbf{A}$ or $\mathbf{B}$ with equal probability and then we apply the chosen device to the qubit according to the usual von-Neumann measurement postulate.

(a) Show that $\langle +|-\rangle = 0$ and that $[\mathbf{A}, \mathbf{B}] \neq 0$. (1P)

(b) Suppose that the system is initially in the state $|\psi\rangle$ and let us apply the random measurement M defined above. Determine the possible post-measurement states and their probability. (1P)

(c) Assume that the system is initially in one of the states $|0\rangle, |1\rangle, |+\rangle, |-\rangle$ with the probabilities p_0, p_1, p_+, p_-, respectively. Apply the mixed measurement M and determine the probabilities after the measurement. (1P)

(d) Starting with the initial state $|\psi\rangle = |0\rangle$, compute p_0, p_1, p_+, p_- after one and two $\mathbf{A}/\mathbf{B}$ measurements. (1P)

(e) Calculate p_0, p_1, p_+, p_- after infinitely many such measurements. Hint: One possible way would be to look at the matrix that maps the probabilities in its spectral representation. (2P)

Exercise 3.3: Trace (2P)

(a) Let $\mathbf{A}$ and $\mathbf{B}$ be two $n \times n$ matrices. Show that $\mathrm{Tr}[\mathbf{AB}] = \mathrm{Tr}[\mathbf{BA}]$.

(b) Prove that the trace is invariant under unitary transformations.

$(\Sigma = \mathbf{12P})$

To be handed in on Monday, November 13, at the beginning of the tutorial.