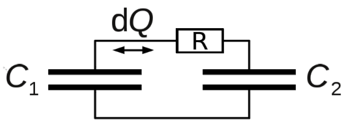
Charged capacitors carry entropy [Czar, Wikimedia]

EXERCISE 2.1: EQUILIBRATION OF TWO CAPACITORS (5P)



The entropy of a charged capacitor is given by

$$H(Q) = const + \frac{Q^2}{2Ck_BT},$$

where Q is the charge, C the capacity, k_B the Boltzmann constant, and T is the constant temperature.

(a) Justify this formula using the Clausius relation between differential work and differential entropy $dW = T\, k_B\, \mathrm{d}H$. (1P)

(b) Let us now consider two capacitors (at the same temperature) with the capacities C_1, C_2 which can exchange electric charges, as indicated in the figure. The total electric charge is a conserved quantity. Which "temperature-like" quantity attains the same value in both capacitors after equilibration and what would be the interpretation of this quantity? (1P)

(c) Consider two identical capacitors ($C_1 = C_2 = C$), one of them charged at voltage $U_1 = U$ and the other one discharged ($U_2 = 0$). If they are coupled as in the figure, they will both equilibrate at half of the voltage, i.e. $U_1 = U_2 = U/2$. Compute the entropy change of the two capacitors before and after equilibration. (1P)

(d) Does the result of (c) violate the Second Law? Think about it and compute the entropy production of the entire system during equilibration. (2P)

EXERCISE 2.2: ALTERNATIVE ENTROPIES OBEYING THE SECOND LAW (7P)

Consider a Markov system with configurations $c \in \Omega$ and given rates $w_{c\to c'} \geq 0$. Suppose that the actual probability distribution at time t is $\{p_c(t)\}$. Furthermore, let

$$f : [0,1] \to \mathbb{R}$$

be some function that map the probabilities to a real number.

(a) Let $\langle f \rangle = \sum_c p_c(t) f\big(p_c(t)\big)$ be the expectation value of f at time t. Compute its temporal derivative using the master equation. (2P)

(b) Show that in the case of a closed system, where the rates are known to be symmetric ($w_{c\to c'} = w_{c'\to c}$), the temporal derivative can be expressed as

$$\frac{\mathrm{d}\langle f\rangle}{\mathrm{d}t} = \frac{1}{2}\sum_{c,c'} w_{c\to c'}\Big(p_{c'}(t) - p_c(t)\Big)\Big(g\big(p_c(t)\big) - g\big(p_{c'}(t)\big)\Big)\,,$$

where $g(p) = f(p) + p\,f'(p)$. (1P)

(c) Find a simple sufficient condition on g which ensures that $\langle f\rangle$ obeys a Second Law, meaning that $\frac{\mathrm{d}}{\mathrm{d}t}\langle f\rangle \geq 0$ and $\frac{\mathrm{d}}{\mathrm{d}t}\langle f\rangle = 0$ if and only if the system is in equilibrium. (1P)

(d) Demonstrate that the Shannon entropy and the Tsallis entropy both obey the Second Law. (2P)

(e) Invent a new entropy (find a function f) that satisfies the Second Law. (1P)

(Σ = 12P)

To be handed in on Monday, October 30, extended until November 06, at the beginning of the tutorial.