Sample Solutions Exercise 10

Exercise 10.1: PPT criterion (4P)

A unitary transformation $\mathbf{U}$ allows one to convert a separable state ρ into an entangled state $\rho' = \mathbf{U}\rho\mathbf{U}^\dagger$ and vice versa. In this context let us consider

$$\rho = \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \qquad \mathbf{U} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}.$$

(a) Apply the PPT criterion to the density matrix before and after the unitary transformation. (2P)

(b) Determine the entanglement of formation E_F according to the Wooters formula (see lecture notes) before and after the transformation. (2P)

Sample Solution

(a) The initial state ρ is invariant under partial transposition, i.e. $\rho^{T_X} = \rho$. Thus the PPT criterion will not apply, meaning that the state is not necessarily entangled. In fact, the state is separable, as can be seen by writing it in the form $\rho = \frac{1}{4}(\mathbb{1} \otimes \mathbb{1} + \sigma^x \otimes \sigma^x)$.
The transformed state

$$\rho' = \mathbf{U}\rho\mathbf{U}^\dagger = \frac{1}{4}\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

is obviously *not* invariant under partial transposition. The reason is that T_X simply exchanges the non-diagonal 2×2 blocks in the matrix, hence we have

$$(\rho')^{T_X} = \frac{1}{4}\begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

This matrix possesses the negativx eigenvalue $\frac{1}{4}(1-\sqrt{2})$, meaning that the PPT criterion signals entanglement.

(b) Computing the entanglement of formation according to Wooters formula we find

$$\Lambda(\rho) = \frac{1}{8}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \qquad \Lambda(\rho') = \frac{1}{8}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

with the spectra $\{\frac{1}{4}, \frac{1}{4}, 0, 0\}$ und $\{\frac{1}{4}, 0, 0, 0\}$. Taking the square roots of the eigenvalues and using the formulas given in the lecture notes one ends up with $C(\rho) = 0$ and $C(\rho') = \frac{1}{2}$. Thus we obtain

$$E_F(\rho) = S[0] = 0\,. \qquad E_F(\rho') = S[\frac{1}{4}(2+\sqrt{3})] \approx 0.354579 \neq 0\,.$$

Exercise 10.2: Polar decomposition (3P)

(a) Use the singular value decomposition to show that any quadratic matrix $\mathbf{A}$ can be written in the form $\mathbf{A} = \mathbf{UP}$, where $\mathbf{U}$ is a unitary matrix while $\mathbf{P}$ is Hermitean and positive definite.

(b) Show that $\mathbf{P} = \sqrt{\mathbf{A}^\dagger \mathbf{A}}$.

(c) Prove that the eigenvalues of a unitary transformations are roots of the unit circle in the complex plane. Use this finding to show that the polar decomposition induces a representation of the determinant of $\mathbf{A}$ in polar coordinates, namely, $\det(\mathbf{A}) = re^{i\phi}$.

Sample Solution

(a) The singular value decomposition of $\mathbf{A}$ reads

$$\mathbf{A} = \mathbf{V}\Sigma\mathbf{W}^\dagger\,,$$

where $\mathbf{V}, \mathbf{W}$ is unitary and Σ is diagonal with non-negative entries on the diagonal. Because of $\mathbf{W}^\dagger\mathbf{W} = \mathbb{1}$ one has

$$\mathbf{A} = \underbrace{\mathbf{V}\mathbf{W}^\dagger}_{=:\mathbf{U}}\underbrace{\mathbf{W}\Sigma\mathbf{W}^\dagger}_{=:\mathbf{P}}\,.$$

Since Σ is Hermitean and positive semidefinite, the same applies to the unitarily transformed operator $\mathbf{P} = \mathbf{W}\Sigma\mathbf{W}^\dagger$ since the eigenvalues and the property of being Hermitean is preserved under unitary transformations.

(b) Taking the result of (a) and plugging in one obtains

$$\mathbf{A}^\dagger\mathbf{A} = \mathbf{P}^\dagger\mathbf{U}^\dagger\mathbf{U}\mathbf{P} = \mathbf{P}^\dagger\mathbf{P} = \mathbf{P}^2\,.$$

Since $\mathbf{P}$ is positive semidefinite, one can directly compute the (positive) square root:

$$\mathbf{P} = \sqrt{\mathbf{A}^\dagger\mathbf{A}}.$$

Alternative derivation:

$$\mathbf{A}^\dagger\mathbf{A} = \mathbf{W}\Sigma^\dagger\mathbf{V}^\dagger\mathbf{V}\Sigma\mathbf{W}^\dagger = \mathbf{W}\Sigma^\dagger\Sigma\mathbf{W}^\dagger = \mathbf{W}\Sigma^2\mathbf{W}^\dagger$$

The square root of a matrix can be defined if it is diagonalizable and positive definite, meaning that all eigenvalues are non-negative. Since the diganonalized version of $\mathbf{A}^\dagger\mathbf{A}$ is just given by Σ^2 which happens to be a diagonal matrix with non-negative entries, we can conclude taht

$$\sqrt{\mathbf{A}^\dagger\mathbf{A}} = \mathbf{W}\sqrt{\Sigma^2}\mathbf{W}^\dagger = \mathbf{W}\Sigma\mathbf{W}^\dagger = \mathbf{P}\,.$$

(c) The eigenvalue problem and corresponding complex conjugated eigenvalues problem of a unitary matrix $\mathbf{U}$ reads

$$\mathbf{U}|\psi\rangle = \lambda\psi\,, \qquad \langle\psi|\mathbf{U}^\dagger = \lambda^*\langle\psi|\,.$$

This implies that

$$\underbrace{\langle\psi|\mathbf{U}^\dagger\mathbf{U}|\psi\rangle}_{=\langle\psi|\psi\rangle=1} = \lambda\lambda^* \underbrace{\langle\psi|\psi\rangle}_{=1},$$

hence $\lambda\lambda^* = |\lambda|^2 = 1$, meaning that the eigenvalue $\lambda = e^{i\phi}$ resides on the complex unit circle. Since

$$1 = \det(\mathbb{1}) = \det(\mathbf{U}^\dagger\mathbf{U}) = \det(\mathbf{U}^\dagger)\det(\mathbf{U}) = \left[\det(\mathbf{U})\right]^*\det(\mathbf{U}) = |\det\mathbf{U}|^2,$$

we can conclude that $\det(\mathbf{U})$ lies on the complex unit circle.

Alternative argument:

$$det\mathbf{A} = \underbrace{det\mathbf{U}}_{=e^{i\phi}}\cdot\underbrace{det\mathbf{P}}_{\in\mathbb{R}_0^+} = re^{i\phi}\,,$$

where the determinant of $\mathbf{U}$ *is given by the product of the eigenvalues* $\prod_j e^{i\phi_j}$. *This implies that the determinant is also located on the unit circle, thus it can be written in the form* $e^{i\phi}$.

Exercise 10.3: Trace distance **(5P)**

Consult the internet to find the definition of the *trace distance* between two density matrices ρ, σ and the so-called *fidelity*.

(a) Compute trace distance of two qubits in the following special cases: (2P)

- $\rho = |0\rangle\langle 0|$ and $\sigma = |+\rangle\langle +|$, where $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.
- $\rho = \frac{1}{4}\begin{pmatrix}1 & 0\\ 0 & 3\end{pmatrix}$ and $\sigma = \frac{1}{4}\begin{pmatrix}1 & 1\\ 1 & 3\end{pmatrix}$.
- $\rho = |\psi\rangle\langle\psi|$ and $\sigma = p|\psi\rangle\langle\psi| + (1-p)\frac{\mathbb{1}}{2}$, where $p \in [0,1]$ and $|\psi\rangle$ is an arbitrary ket state.

(b) Compute the fidelity for the cases listed above. (2P)

(c) Verify that the inequality

$$1 - F \leq T \leq \sqrt{1 - F^2}$$

holds in all cases. (1P)

Sample Solution

(a) The trace distance is defined as

$$T(\rho,\sigma) = \frac{1}{2}\mathrm{Tr}\left[\sqrt{(\rho-\sigma)^\dagger(\rho-\sigma)}\right]$$

which is most easily understood in a basis where $\rho-\sigma$ is diagonal. We compute this quantity for the following three cases:

- $\rho = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \sigma = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \Rightarrow \quad (\rho-\sigma)^\dagger(\rho-\sigma) = \frac{\mathbb{1}}{2} \quad \Rightarrow \quad T = \frac{1}{\sqrt{2}}.$
- $\rho-\sigma = \frac{1}{4}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \quad \Rightarrow \quad (\rho-\sigma)^\dagger(\rho-\sigma) = \frac{\mathbb{1}}{16} \quad \Rightarrow \quad T = \frac{1}{4}.$
- $\rho-\sigma = (1-p)|\psi\rangle\langle\psi| - (1-p)\frac{\mathbb{1}}{2}$
  $\Rightarrow \quad (\rho-\sigma)^\dagger(\rho-\sigma) = (1-p)^2(1-\frac{1}{2}-\frac{1}{2})|\psi\rangle\langle\psi| + (1-p)^2\frac{\mathbb{1}}{4}.$
  $\Rightarrow \quad T = \frac{1-p}{2}.$

(b) The fidelity is defined by

$$F(\rho,\sigma) = \mathrm{Tr}\left[\sqrt{\sqrt{\rho}\,\sigma\sqrt{\rho}}\right].$$

- In this case we have $\rho^2 = \rho \Rightarrow \sqrt{\rho} = \rho$. It follows that $\sqrt{\rho}\sigma\sqrt{\rho} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix}$ also $F = \frac{1}{\sqrt{2}}$.
- Here we have $\sqrt{\rho} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{3} \end{pmatrix} \quad \rightarrow \quad \sqrt{\rho}\sigma\sqrt{\rho} = \frac{1}{16}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 9 \end{pmatrix}.$

  This matrix has the eigenvalues $\frac{1}{16}(5\pm\sqrt{19})$.
  Hence $F = \frac{1}{4}(\sqrt{5-\sqrt{19}} + \sqrt{5+\sqrt{19}}) \approx 0.96498$.
- The square root of a projector is always the projector itself, hence $\sqrt{\rho}\sigma\sqrt{\rho} = \rho\sigma\rho = \frac{1+p}{2}|\psi\rangle\langle\psi|$. The trace of (the square root of) a projector equals 1, leading to $F = \sqrt{\frac{1+p}{2}}$.

(c) In the first two cases the inequality is trivial to check. In the third case we have to prove that

$$1 - \sqrt{\frac{1+p}{2}} \le \frac{1-p}{2} \le \sqrt{1 - \frac{1-p}{2}}.$$

The first inequality is equivalent to

$$\sqrt{\frac{1+p}{2}} \ge \frac{1+p}{2}$$

which is obviously true on $p \in [0,1]$. For showing the second inequality we note that all terms are nonnegative so that the whole inequality may be squared:

$$\frac{p^2}{4} - \frac{p}{2} + \frac{1}{4} \le \frac{1}{2} - \frac{p}{2}$$

Adding $p/2 - 1/4$ this turns into

$$\frac{p^2}{4} \le \frac{1}{4}$$

which also holds on $p \in [0,1]$.
