Sample Solutions Exercise 8

Exercise 8.1: GHZ state (3P)

The GHZ state for a three-qubit system with $\mathcal{H} = (\mathbb{C}^2)^{\otimes 3} = \mathbb{C}^8$ is defined by

$$|GHZ\rangle = \frac{1}{\sqrt{2}}\Big(|000\rangle + |111\rangle\Big).$$

(a) Compute the reduced 2×2 density matrix of the third qubit and find out whether this qubit is entangled with the other two qubits. (1P)

(b) Calculate the reduced 4×4 density matrix of the first two qubits and calculate the corresponding von-Neumann entropy. (1P)

(c) Is the state determined in (b) separable or entangled? (1P)

Sample Solution

The GHZ state ρ_{GHZ} is a pure state. In the standard qubit configuration basis it is given by a matrix with nonzero entries in the four corners:

$$\rho_{123}^{GHZ} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(a) Tracing out the first two qubits one ends up with the 2×2 matrix

$$\rho_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Obviously, the entropy of the reduced state is 1, meaning that this state is not pure. As the state of the total system is still pure, this means that the third bit is entangled with the other two qubits.

(b) Carrying out the partial trace over the third qubit one obtains the 2-qubit state

$$\rho_{12} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The entropy of this state is 1, hence we are dealing here again with a mixed sate. However, the state has no non-diagonal entries and it is in fact separable:

$$\rho_{12} = \frac{1}{2}\Big(|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 1|\Big).$$

(c) The result of (b) tells us that the two qubits are only *classically* correlated but non-entangled. This is a special property of GHZ states: Tracing out one of the components destroys the entanglement between the other components, converting it into classical correlation.

Exercise 8.2: Creatioin of entanglement by measurement (4P)

Consider the pure 4-qubit state

$$|\Psi_\alpha\rangle = |\psi_\alpha\rangle \otimes |\psi_\alpha\rangle \qquad \text{with} \quad |\psi_\alpha\rangle = \alpha|00\rangle + \sqrt{1-\alpha^2}|11\rangle.$$

Suppose that we perform a projective measurement with the measurement operator

$$\mathbf{M} \;=\; \sigma^z \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} + \mathbb{1} \otimes \mathbb{1} \otimes \sigma^z \otimes \mathbb{1}.$$

The purpose of this exercise is to show that measurements do not always destroy entanglement (converting it into classical correlations of outcomes), but measurements can also be used to create entanglement.

(a) Compute the eigenvalues of $\mathbf{M}$ and their degeneracies. (1P)

(b) Determine the projection operators of the measurement. (1P)

(c) In which state is the system after the measurement given that the measurement result is known? (1P)

(d) What is the probability to obtain a fully entangled state between the two left and the two right qubits? (1P)

Sample Solution

The initial state reads:

$$|\Psi_\alpha\rangle \;=\; \alpha^2|0000\rangle + \alpha\beta|0011\rangle + \alpha\beta|1100\rangle + \beta^2|1111\rangle\,,$$

where $\beta = \sqrt{1-\alpha^2}$.

(a) The eigenvalues are $+2$ (4-fold degenerate), 0 (8-fold degenerate) and -2 (4-fold degenerate).

(b) The corresponding projectors read

$$\begin{aligned}
\Pi_{+2} &= |0\rangle\langle 0| \otimes \mathbb{1} \otimes |0\rangle\langle 0| \otimes \mathbb{1} \\
\Pi_{0} &= |0\rangle\langle 0| \otimes \mathbb{1} \otimes |1\rangle\langle 1| \otimes \mathbb{1} \;+\; |1\rangle\langle 1| \otimes \mathbb{1} \otimes |0\rangle\langle 0| \otimes \mathbb{1} \\
\Pi_{-2} &= |1\rangle\langle 1| \otimes \mathbb{1} \otimes |1\rangle\langle 1| \otimes \mathbb{1}
\end{aligned}$$

(c) After the measurement the system is (depending on the outcome) in the following state:

$$\begin{aligned}
+2: &\quad |0000\rangle \\
0: &\quad \frac{1}{\sqrt{2}}\Big(|0011\rangle + |1100\rangle\Big) \\
-2: &\quad |1111\rangle
\end{aligned}$$

Note that the coefficients α, β drop out due to normalization.

(d) As one can see, the only state which is maximally entangled is the state for the measurement result 0. The probability to be in this state is

$$\langle\psi_\alpha|\Pi_0|\psi_\alpha\rangle \;=\; 2|\alpha\beta|^2.$$

EXERCISE 8.3: SCHMIDT DECOMPOSITION **(5P)**

Determine the Schmidt decomposition of the following 2-qubit states (please sketch your calculation step by step):

(a) $|\psi\rangle = \frac{1}{2}\Big(|00\rangle + |01\rangle + |10\rangle + |11\rangle\Big)$. (1P)

(b) $|\psi\rangle = \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle\Big)$. (1P)

(c) $|\psi\rangle = \frac{1}{2}\Big(|00\rangle + |01\rangle + 2|10\rangle\Big)$. (3P)

SAMPLE SOLUTION

(a) The Schmidt decomposition can be computed according to the recipe sketched in the lecture notes:

$$\psi_{ij} = \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} \quad\Rightarrow\quad \mathbf{M} = \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} \quad\Rightarrow\quad \mathbf{M}^\dagger\mathbf{M} = \mathbf{M}\mathbf{M}^\dagger = \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}.$$

This matrix has the eigenvalues 0 and 1, hence $r = 1$ and the only Schmidt number is $\alpha_1 = 1$. Thus we have actually a product state of the form $|\psi\rangle = \alpha_1|\phi_1\rangle_X \otimes |\phi_1\rangle_Y$ with $|\phi_1\rangle_X = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$ and $|\phi_1\rangle_Y = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$.

(b) For the Bell state $|\psi\rangle = \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle\Big)$ we have

$$\psi_{ij} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \quad\Rightarrow\quad \mathbf{M} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \quad\Rightarrow\quad \mathbf{M}^\dagger\mathbf{M} = \mathbf{M}\mathbf{M}^\dagger = \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

Hence in this case we have two non-vanishing Schmidt numbers $\alpha_1 = \alpha_2 = 1/\sqrt{2}$. Since the eigenvalues are degenerate, the corresponding (orthonormal) eigenvectors can be chosen freely. The easiest choice is to take just the qubit basis vectors, i.e.,

$$|\phi_1\rangle_{X,Y} = \begin{pmatrix}1\\0\end{pmatrix} = |0\rangle_{X,Y}\,, \quad |\phi_1\rangle_{X,Y} = \begin{pmatrix}0\\1\end{pmatrix} = |1\rangle_{X,Y}\,.$$

and therefore we get

$$|\psi\rangle = \frac{1}{\sqrt{2}}\Big(|0\rangle_X \otimes |0\rangle_Y + |1\rangle_X \otimes |1\rangle Y\Big)\,.$$

This is of course the literally the same state that was given in the beginning.

Alternative: Therefore, an alternative short solution of this part would be to recognize that the given Bell state $|\psi\rangle = \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle\Big)$ is already written down in a Schmidt form (because there are no mixed qubit states $|01\rangle$ and $|10\rangle$.

(c) For the third state $|\psi\rangle = \frac{1}{2}\Big(|00\rangle + |01\rangle + 2|10\rangle\Big)$ we really have to do the calculation:

$$\psi_{ij} = \frac{1}{2}\begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \quad \Rightarrow \quad \mathbf{M} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} \quad \Rightarrow \quad \mathbf{M}^\dagger\mathbf{M} = \frac{1}{4}\begin{pmatrix} 5 & 1 \\ 1 & 1 \end{pmatrix}, \quad \mathbf{M}\mathbf{M}^\dagger = \frac{1}{4}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

The square roots of the eigenvalues (Schmidt numbers) are given by

$$\alpha_{1,2} = \frac{1}{2}\sqrt{3 \pm \sqrt{5}}$$

The left Schmidt vectors $|\phi_i\rangle_X$ are the (complex conjugate of the) eigenvectors of $\mathbf{M}^\dagger\mathbf{M}$. *Mathematica*® tells us that these eigenvectors are given by $(2 \pm \sqrt{5}, 1)$, but these eigenvectors still have to be normalized. The result reads

$$|\phi_1\rangle_X = \frac{1}{\sqrt{10+4\sqrt{5}}}\begin{pmatrix} 2+\sqrt{5} \\ 1 \end{pmatrix}, \quad |\phi_2\rangle_X = \frac{1}{\sqrt{10-4\sqrt{5}}}\begin{pmatrix} 2-\sqrt{5} \\ 1 \end{pmatrix}$$

The right Schmidt vectors $|\phi_i\rangle_Y$ are the eigenvectors of $\mathbf{M}\mathbf{M}^\dagger$:

$$|\phi_1\rangle_Y = \sqrt{\frac{5+\sqrt{5}}{40}}\begin{pmatrix} +\sqrt{5}-1 \\ 2 \end{pmatrix}, \quad |\phi_2\rangle_Y = \sqrt{\frac{5-\sqrt{5}}{40}}\begin{pmatrix} -\sqrt{5}-1 \\ 2 \end{pmatrix}$$

With these vectors we can verify that

$$|\psi\rangle = \alpha_1|\phi_1\rangle_X \otimes |\phi_1\rangle_Y + \alpha_2|\phi_2\rangle_X \otimes |\phi_2\rangle_Y$$

(Σ = 12P)