SAMPLE SOLUTIONS EXERCISE 6

EXERCISE 6.1: DETERMINANT OF A TENSOR PRODUCT (2P)

Let $\mathbf{A}$ and $\mathbf{B}$ two finite-dimensional square matrices of dimensions $n \times n$ and $m \times m$. Express $\det(\mathbf{A} \otimes \mathbf{B})$ in terms of $\det(\mathbf{A})$ and $\det(\mathbf{B})$ and prove your result in such a way that it even holds for non-diagonalizable matrices.

SAMPLE SOLUTION

First let us assume that both matrices are diagonalizable. Let α_i be the eigenvalues of $\mathbf{A}$ and $|e_i\rangle$ the corresponding eigenvectors. Likewise, let β_j be the eigenvalues of $\mathbf{B}$ and $|f_j\rangle$ the corresponding eigenvectors. Then we can define a basis $|g_{ij}\rangle = |e_i\rangle \otimes |f_j\rangle$ so that the matrix of $\mathbf{C} = \mathbf{A} \otimes \mathbf{B}$ is diagonal. In this basis we can compute the determinant (which is basis-independent):

$$\langle g_{ij}|\mathbf{C}|g_{kl}\rangle = \langle g_{ij}|(\mathbf{A} \otimes \mathbf{B})|g_{kl}\rangle = \langle e_i|\mathbf{A}|e_k\rangle \langle f_j|\mathbf{B}|f_l\rangle = \alpha_i \delta_{ik} \beta_j \delta_{jl} \,.$$

$$\Rightarrow \quad \det(\mathbf{C}) = \prod_{ij} \alpha_i \beta_j = \Big(\prod_{ij} \alpha_i\Big)\Big(\prod_{ij} \beta_j\Big)$$

$$= \Big(\prod_j \det(\mathbf{A})\Big)\Big(\prod_i \det(\mathbf{B})\Big) = \det(\mathbf{A})^m \det(\mathbf{B})^n.$$

In the case of non-diagonalizable matrices chose basis systems $|e_i\rangle$ and $|f_j\rangle$ such that $\mathbf{A}$ and $\mathbf{B}$ are represented in a Jordan normal form, meaning that the diagonal contains the eigenvalues and that there are no non-zero entries below the diagonal. By construction, the same applies to the tensor product. Hence the determinant is again given by the product of the eigenvalues so that the proof given above remains valid.

EXERCISE 6.2: THE PAULI BASIS (6P)

Let us consider standard Pauli matrices including the identity

$$\sigma_0 = \mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and let us define an N-qubit Pauli operator by the tensor product

$$\sigma_{\mathbf{j}} = \sigma_{j_1} \otimes \sigma_{j_2} \otimes \cdots \otimes \sigma_{j_N} = \bigotimes_{i=1}^{N} \sigma_{j_i}.$$

where $\mathbf{j} = (j_1, j_2, \ldots, j_N)$ is a N-component multiindex with $j_1, \ldots, j_n \in \{0, 1, 2, 3\}$.

(a) Show that $\mathrm{Tr}[\sigma_{\mathbf{j}}\sigma_{\mathbf{k}}] = 2^N \delta_{\mathbf{j},\mathbf{k}}$. (2P)

(b) Let $\mathbf{A}$ be an operator acting on the Hilbert space $\mathbb{C}^{2^N}$. Show that this operator can always be represented by $\mathbf{A} = \sum_{\mathbf{j}} A_{\mathbf{j}}\sigma_{\mathbf{j}}$ and calculate the coefficients $A_{\mathbf{j}}$. (2P)

(c) How are Hermitean operators ($\mathbf{A} = \mathbf{A}^\dagger$) reflected in the components $A_{\mathbf{j}}$? (1P)

(d) Express the scalar product $\langle \vec{A}|\vec{B}\rangle = \sum_{\mathbf{j}} A_{\mathbf{j}}^* B_{\mathbf{j}}$ in terms of $\mathbf{A}$ and $\mathbf{B}$. (1P)

Sample Solution

(a) For individual Pauli matrices we have

$$\mathrm{Tr}[\sigma_j\sigma_k] = 2\delta_{j,k}.$$

For Pauli tensor strings the same happens independently in each of the tensor slots:

$$\mathrm{Tr}[\sigma_{\mathbf{j}}\sigma_{\mathbf{k}}] = \mathrm{Tr}\Big[\Big(\bigotimes_{i=1}^N \sigma_{j_i}\Big)\Big(\bigotimes_{i=1}^N \sigma_{k_i}\Big)\Big] = \mathrm{Tr}\Big[\bigotimes_{i=1}^N \big(\sigma_{j_i}\sigma_{k_i}\big)\Big]$$

$$= \prod_{i=1}^N \mathrm{Tr}[\sigma_{j_i}\sigma_{k_i}] = \prod_{i=1}^N 2\delta_{j_i,k_k} = 2^N \prod_{i=1}^N \delta_{j_i,k_k} = 2^N\delta_{\mathbf{j},\mathbf{k}}.$$

(b) We simply use the orthogonality relation derived in (a):

$$\mathrm{Tr}[\mathbf{A}\sigma_{\mathbf{k}}] = \mathrm{Tr}\Big[\Big(\sum_{\mathbf{j}} A_{\mathbf{j}}\sigma_{\mathbf{j}}\Big)\sigma_{\mathbf{k}}\Big] = \sum_{\mathbf{j}} A_{\mathbf{j}}\,\mathrm{Tr}[\sigma_{\mathbf{j}}\sigma_{\mathbf{k}}] = 2^N\sum_{\mathbf{j}} A_{\mathbf{j}}\delta_{\mathbf{j},\mathbf{k}} = 2^N A_{\mathbf{k}}.$$

$$\Rightarrow \quad A_{\mathbf{k}} = \frac{1}{2^N}\mathrm{Tr}[\mathbf{A}\sigma_{\mathbf{k}}]$$

(c) Since all Pauli operators are Hermitean, we have

$$A_{\mathbf{k}}^* = \frac{1}{2^N}\mathrm{Tr}[\sigma_{\mathbf{k}}^\dagger\mathbf{A}^\dagger] = \frac{1}{2^N}\mathrm{Tr}[\mathbf{A}^\dagger\sigma_{\mathbf{k}}^\dagger] = \frac{1}{2^N}\mathrm{Tr}[\mathbf{A}\sigma_{\mathbf{k}}] = A_{\mathbf{k}}.$$

Therefore, Hermitean matrices are represented by real-valued components $A_{\mathbf{k}} \in \mathbb{R}$.

(d) Consider the product

$$\mathbf{A}^\dagger\mathbf{B} = \Big(\sum_{\mathbf{j}} A_{\mathbf{j}}^*\sigma_{\mathbf{j}}\Big)\Big(\sum_{\mathbf{k}} B_{\mathbf{k}}\sigma_{\mathbf{k}}\Big) = \sum_{\mathbf{j},\mathbf{k}} A_{\mathbf{j}}^* B_{\mathbf{k}}\sigma_{\mathbf{j}}\sigma_{\mathbf{k}}$$

$$\Rightarrow \quad \mathrm{Tr}\Big[\mathbf{A}^\dagger\mathbf{B}\Big] = \sum_{\mathbf{j},\mathbf{k}} A_{\mathbf{j}}^* B_{\mathbf{k}}\underbrace{\mathrm{Tr}[\sigma_{\mathbf{j}}\sigma_{\mathbf{k}}]}_{\delta_{\mathbf{j},\mathbf{k}}} = \sum_{\mathbf{j}} A_{\mathbf{j}}^* B_{\mathbf{j}} = \langle \vec{A}|\vec{B}\rangle.$$

Exercise 6.3: Entanglement of a pure 2-qubit state (4P)

Consider the 2-qubit state $\rho = |\psi\rangle\langle\psi|$ with

$$|\psi\rangle = \frac{1}{5}|00\rangle + \frac{2i\sqrt{2}}{5}|01\rangle - \frac{4}{15}|10\rangle - \frac{8i\sqrt{2}}{15}|11\rangle.$$

(a) Use *Mathematica*® or similar tools to compute the 4×4 matrix ρ in the qubit basis. Verify that ρ is pure. (2P)

(b) Compute the reduced density matrices $\rho^{(L)}$ und $\rho^{(R)}$ for the left and the right qubit by carrying out the partial traces (1P)

$$\rho^{(L)}_{jk} = \sum_{m=0}^{1} \rho_{jm,km}\,, \qquad \rho^{(R)}_{mn} = \sum_{j=0}^{1} \rho_{jm,jn}.$$

(c) A pure state is called *entangled* if the reduced density matrices are mixed. Find out whether ρ is entangled or not. (1P)

Sample Solution

(a) The result reads

$$\rho_{jm,kn} = \langle jm|\psi\rangle\langle\psi|kn\rangle = \psi_{j,m}\psi^*_{k,n} = \begin{pmatrix} +\frac{1}{25} & -\frac{2i\sqrt{2}}{25} & -\frac{4}{75} & +\frac{8i\sqrt{2}}{75} \\ +\frac{2i\sqrt{2}}{25} & +\frac{8}{25} & -\frac{8i\sqrt{2}}{75} & -\frac{32}{75} \\ -\frac{4}{75} & +\frac{8i\sqrt{2}}{75} & +\frac{16}{225} & -\frac{32i\sqrt{2}}{225} \\ -\frac{8i\sqrt{2}}{75} & -\frac{32}{75} & +\frac{32i\sqrt{2}}{225} & +\frac{128}{225} \end{pmatrix}$$

where we assumed the canonical qubit basis $|00\rangle, |01\rangle, |10\rangle, |11\rangle$. There are several ways to test purity. One of them is to determine the eigenvalues with *Mathematica*® and to see that they are $\{0, 0, 0, 1\}$, hence the entropy is zero. Another way of testing it is to verify that $\rho^2 = \rho$, meaning that ρ is a projector and hence the state is pure.

(b) The result reads:

$$\rho^{(L)} = \begin{pmatrix} \frac{9}{25} & -\frac{12}{25} \\ -\frac{12}{25} & \frac{16}{25} \end{pmatrix}, \qquad \rho^{(R)} = \begin{pmatrix} \frac{1}{9} & -\frac{2i\sqrt{2}}{9} \\ \frac{2i\sqrt{2}}{9} & \frac{8}{9} \end{pmatrix}$$

(c) We can test purity in the same way as in (a). It turns out that the reduced states are pure. Thus, the two qubits are not entangled. This shows that "complicated-looking amplitudes" do not automatically imply entanglement.

(Σ = 12P)