Quantum Information Theory

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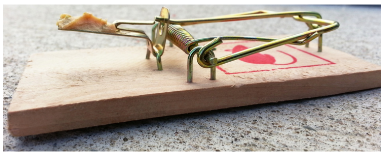


Free energy can be dangerous.

Exercise 12.1: Modular Hamiltonian and Free Energy (3P)

For a given density operator σ, the so-called *modular Hamiltonian* $\mathbf{H}(\rho)$ is the operator which would have generated it, i.e.,

$$\sigma = \exp\big(-\mathbf{H}(\sigma)\big).$$

Let σ be a density matrix which represents a certain thermodynamic equilibrium state. Then the corresponding *generalized free energy* F_σ is defined as

$$F_\sigma(\rho) \;:=\; \langle \mathbf{H}(\sigma)\rangle_\rho - S(\rho).$$

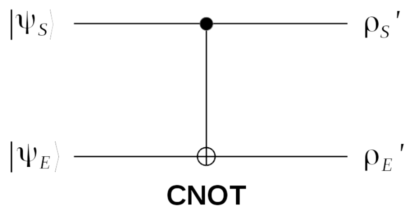
where $S(\rho)$ is the usual von-Neumann entropy.

(a) Show that the generalized free energy vanishes in equilibrium, i.e., if $\rho = \sigma$. (1P)

(b) Show that $F_\sigma(\rho) > 0$ for all $\rho \neq \sigma$, proving that F is minimal at equilibrium. *Hint: Relate it to the relative quantum entropy.* (2P)

Exercise 12.2: CNOT Gate (9P)

Suppose that a system S is prepared in a pure 1-qubit state $|\psi_S\rangle = \cos\theta\,|0_S\rangle + \sin\theta\,|1_S\rangle$. Assume that system is in contact with an environment E, which for simplicity is modeled here as a single qubit that is initially in the pure state $|\psi_E\rangle = |0_E\rangle$. The system interacts with the environment by means of a CNOT gate, as shown in the figure on the right side.



(a) Go to the literature in order to get familiar with the CNOT gate and write down the corresponding unitary transformation on the total system in matrix form as well as in Dirac notation. Let us use the convention that the left tensor slot represents the system while the right tensor factor represents the environment. (1P)

(b) Show that the interaction via CNOT puts the system into the mixed state (2P)

$$\rho'_S = \cos^2\theta\,|0\rangle\langle 0|_S + \sin^2\theta\,|1\rangle\langle 1|_S\,.$$

(c) Use the spectral decomposition to demonstrate that in arbitrary finite-dimensional Hilbert spaces any unitary operator $\mathbf{U}$ maps a general product state $\rho = \rho_S \otimes \rho_E$ to a new density matrix $\rho' = \mathbf{U}\rho\mathbf{U}^{-1}$ in such a way that the reduced density matrix of the system can be written in the Kraus form (2P)

$$\rho'_S = \sum_\alpha \mathbf{B}_\alpha \rho_S \mathbf{B}_\alpha^\dagger\,.$$

Please turn over ⇒

(d) Prove that the operators $\mathbf{B}_\alpha$ from part (c) obey the normalization condition (2P)

$$\sum_\alpha \mathbf{B}_\alpha^\dagger \mathbf{B}_\alpha = \mathbb{1}_S \,.$$

(e) Determine the operators $\mathbf{B}_\alpha$ in the special case of a CNOT gate as defined above. (1P)

(f) The so-called *quantum fidelity* of two density matrices ρ_1, ρ_2 acting on the same Hilbert space is defined by

$$F(\rho_1, \rho_2) = \mathrm{Tr}\left[\sqrt{\sqrt{\rho_1}\,\rho_2\,\sqrt{\rho_1}}\right] .$$

Compute $F(\rho_S, \rho'_S)$ by exploiting the fact that ρ_S is a pure state. For which angle θ does the function F become maximal? (1P)

(Σ = 12P)

Last exercise! You are done!
To be handed in on Monday, January 29, at the beginning of the tutorial.