> Theorem 4.1. *Any completely positive ultraweakly continuous linear mapping T of $\mathfrak{A}$ into itself with $\|TB\| \leqslant \|B\|$ is of the form $TB = \sum_{k \in K} A_k{}^* B A_k$ with $\sum_{k \in K} A_k{}^* A_k \leqslant 1$ and all A_k satisfying* (4.5).
>
> Since nothing crucial will be derived here from this theorem, we will omit the proof. The method of proof is the same as for Theorem 3.3, but the proof becomes slightly more involved since one has to investigate the most general representation of the algebra $\mathfrak{A}$ which replaces the algebra $\mathscr{B}(\mathfrak{H})$ considered before.

Karl Kraus himself considered the proof as not particularly important.

Exercise 11.1: Stinespring representation of a measurement (3P)

A qubit in the state ρ is measured by a projective measurement with the operator σ_z.

(a) Find a Stinespring representation, i.e., find a suitably extended Hilbert space together with a unitary transformation $\mathbf{U}$ which resembles the measurement in the sense of the Stinespring theorem. (2P)

(b) Verify that the action of $\mathbf{U}$, after tracing out the ancilla space, reproduces the measurement. (1P)

Exercise 11.2: Kraus and Stinespring representation (9P)

Consider the following quantum operation (known as depolarizing map)

$$\Phi[\rho] \;=\; \frac{p}{n}\mathrm{Tr}[\rho]\,\mathbb{1} \;+\; (1-p)\rho\,, \qquad p \in [0,1]\,, \quad n = \dim \mathcal{H}\,.$$

Let $\{|i\rangle\}$ be an orthonormal basis of $\mathcal{H}$. The purpose of this exercise is to determine a Kraus and a Stinespring representation of this quantum operation.

(a) Consider first the special case p=1 and guess the corresponding Kraus representation. (1P)

(b) Following the notation in the lecture notes (proof of Kraus theorem) compute the eigenvalues λ_k and the eigenvectors $|e_k\rangle$ of the matrix

$$\tilde{\mathbf{A}} \;=\; \Big(\Phi \otimes \mathbb{1}\Big)(\mathbf{A}) \;=\; \sum_{i,j=1}^{n} \Phi\Big(|i\rangle\langle j|\Big) \otimes |i\rangle\langle j|$$

for general p and n. (2P)

(c) Compute the Kraus operators for general n. (2P)

(d) Write these operators as explicit matrices for the special case of a qubit $n = 2$ and verify that the Kraus representation reproduces $\Phi[\rho]$. (2P)

(e) Use the Kraus matrices obtained in (d) to find a Stinespring representation $\mathbf{U}$ in the special case of a qubit $n = 2$. (2P)

(Σ = 12P)

To be handed in on Monday, January 22, at the beginning of the tutorial.