EXERCISE 10.1: PPT CRITERION (4P)

A unitary transformation $\mathbf{U}$ allows one to convert a separable state ρ into an entangled state $\rho' = \mathbf{U}\rho\mathbf{U}^\dagger$ and vice versa. In this context let us consider

$$\rho = \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \qquad \mathbf{U} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}.$$

(a) Apply the PPT criterion to the density matrix before and after the unitary transformation. (2P)

(b) Determine the entanglement of formation E_F according to the Wooters formula (see lecture notes) before and after the transformation. (2P)

EXERCISE 10.2: POLAR DECOMPOSITION (3P)

(a) Use the singular value decomposition to show that any quadratic matrix $\mathbf{A}$ can be written in the form $\mathbf{A} = \mathbf{UP}$, where $\mathbf{U}$ is a unitary matrix while $\mathbf{P}$ is Hermitean and positive definite.

(b) Show that $\mathbf{P} = \sqrt{\mathbf{A}^\dagger\mathbf{A}}$.

(c) Prove that the eigenvalues of a unitary transformations are roots of the unit circle in the complex plane. Use this finding to show that the polar decomposition induces a representation of the determinant of $\mathbf{A}$ in polar coordinates, namely, $\det(\mathbf{A}) = re^{i\phi}$.

EXERCISE 10.3: TRACE DISTANCE (5P)

Consult the internet to find the definition of the *trace distance* between two density matrices ρ, σ and the so-called *fidelity*.

(a) Compute trace distance of two qubits in the following special cases: (2P)

- $\rho = |0\rangle\langle 0|$ and $\sigma = |+\rangle\langle +|$, where $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.
- $\rho = \frac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$ and $\sigma = \frac{1}{4}\begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}$.
- $\rho = |\psi\rangle\langle\psi|$ and $\sigma = p|\psi\rangle\langle\psi| + (1-p)\frac{\mathbb{1}}{2}$, where $p \in [0,1]$ and $|\psi\rangle$ is an arbitrary ket state.

(b) Compute the fidelity for the cases listed above. (2P)

(c) Verify that the inequality

$$1 - F \le T \le \sqrt{1 - F^2}$$

holds in all cases. (1P)

To be handed in on Monday, January 15, at the beginning of the tutorial.