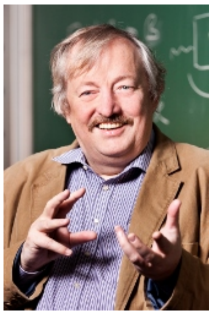
Exercise 9.1: Werner states (6P)



In this exercise we will show that not all non-local quantum correlations are due to entanglement in the sense of non-separability. To this end consider a two-qubit system in the state

$$\rho_z := \frac{1-z}{4}\mathbb{1} + z|\psi\rangle\langle\psi|\,,$$

where $0 \leq z \leq 1$ and $|\psi\rangle = \frac{|01\rangle-|10\rangle}{\sqrt{2}}$. This is a so-called *Werner state*, named after R. Werner at the University of Hannover.

(a) Show that ρ_z is invariant under $SU(2)$, i.e., $\rho_z = (\mathbf{U}^\dagger \otimes \mathbf{U}^\dagger)\rho_z(\mathbf{U}\otimes\mathbf{U})$ for any $\mathbf{U} \in \mathrm{SU}(2)$. Hint: The generators of $SU(2)$ are $T_i = \frac{1}{2}(\sigma_i \otimes \mathbb{1} + \mathbb{1}\otimes\sigma_i)$. (1P)

(b) Show that

$$\rho_z = \frac{1}{6}\sum_{i=1}^{3}\sum_{j=1}^{2}\frac{\mathbb{1}+x_j\sigma_i}{2}\otimes\frac{\mathbb{1}-x_j\sigma_i}{2},$$

where $x_j = \sqrt{3z}(-1)^{j+1}$. Hint: At first calculate the second sum. (2P)

(c) Explain why the expression given in (b) is a physically valid convex decomposition into separable states if $z \leq \frac{1}{3}$ and why this decomposition fails for $z > \frac{1}{3}$. (1P)

(d) A local projective measurement (LPM) on the second qubit is described by

$$\rho_z \to \rho_z' = \sum_{k=1}^{2}(\mathbb{1}\otimes\Pi_k)\rho_z(\mathbb{1}\otimes\Pi_k),$$

where $\Pi_k^2 = \Pi_k$, $\mathrm{Tr}\Pi_k = 1$ and $\mathrm{Tr}\Pi_k\Pi_l = \delta_{kl}$. Choose your favorite Π_1, Π_2 and show that $\rho_z' \neq \rho_z$ for $z > 0$. (1P)

(e) Explain why (a) implies that the inequality $\rho_z' \neq \rho_z$ is independent of your particular choice of $\{\Pi_k\}$. Hence, any LPM will disturb ρ_z even though ρ_z is separable for $z \leq \frac{1}{3}$. (1P)

Exercise 9.2: Quantum discord (6P)

In 2002, Ollivier and Zurek succeeded in constructing a measure for quantum correlations of the type occurring in Werner states. This so-called *quantum discord* D is constructed as follows:

- Consider a bipartite Hilbert space $\mathcal{H} = \mathcal{H}_\mathbf{X}\otimes\mathcal{H}_\mathbf{Y}$.
- When carrying out a local projective measurement (LPM) on $\mathcal{H}_\mathbf{Y}$, as described by a set of projectors $\{\Pi_i\}$, an initial state ρ will change to $\rho|_{\Pi_i} := \frac{(\mathbb{1}\otimes\Pi_i)\rho(\mathbb{1}\otimes\Pi_i)}{\mathrm{Tr}[(\mathbb{1}\otimes\Pi_i)\rho]}$ with probability $p_i = \mathrm{Tr}\,[(\mathbb{1}\otimes\Pi_i)\rho]$.

Please turn over ⇒

- Define $S(\rho_{\mathbf{X}}|\{\Pi_i\}) := \sum_i p_i S(\mathrm{Tr}_{\mathbf{Y}}[\rho|_{\Pi_i}])$. This is the remaining uncertainty about the precise form of $\rho_{\mathbf{X}} = \mathrm{Tr}_{\mathbf{Y}}[\rho]$ after performing the LPM.

$\Rightarrow$ $J(\rho_{\mathbf{X}}|\mathbf{Y}) := S(\rho_{\mathbf{X}}) - \inf_{\{\Pi_i\}} S(\rho_{\mathbf{X}}|\{\Pi_i\})$ is the maximum amount of information we can get about $\rho_{\mathbf{X}}$ with LPMs on $\mathbf{Y}$, i.e., it measures the *classical* correlations of ρ.

- As seen in the lecture, the quantum mutual information $I(\rho)$ captures both *classical and quantum* correlations of ρ, hence we can get the correlations which are purely quantum by calculating the difference:

$$\boxed{D_\rho(\mathbf{X}:\mathbf{Y}) := I(\rho) - J(\rho_{\mathbf{X}}|\mathbf{Y}) = S(\rho_{\mathbf{Y}}) - S(\rho) + \inf_{\{\Pi_i\}} S(\rho_{\mathbf{X}}|\{\Pi_i\}),} \tag{1}$$

where the infimum (minimum) is taken over all possible LPMs $\{\Pi_i\}$. Note that this definition is not necessarily symmetric, but it can be shown that D is non-negative.

In their paper, Ollivier and Zurek showed that $D_\rho(\mathbf{X}:\mathbf{Y}) = 0$ if and only if there exists an LPM on $\mathbf{Y}$ that leaves ρ undisturbed, hence we expect $D_{\rho_z}(\mathbf{X}:\mathbf{Y}) > 0$ for Werner states ρ_z with $z > 0$. The aim of this exercise is to verify this claim.

(a) Show that $S(\rho_z) = 2\log 2 - \frac{1+3z}{4}\log(1+3z) - \frac{3-3z}{4}\log(1-z)$. (2P)

(b) Compute the reduced density matrices $(\rho_z)_{\mathbf{X}} = \mathrm{Tr}_{\mathbf{Y}}[\rho_z]$, $(\rho_z)_{\mathbf{Y}} = \mathrm{Tr}_{\mathbf{X}}[\rho_z]$, the corresponding marginal entropies $S((\rho_z)_{\mathbf{X}})$, $S((\rho_z)_{\mathbf{Y}})$, and show that the inequality

$$S(\rho_z) \geq \max\{S((\rho_z)_{\mathbf{X}}), S((\rho_z)_{\mathbf{Y}})\}$$

holds for all $z \leq \frac{1}{3}$, hence the quantum nature of ρ_z is not revealed by the total and marginal entropies. (1P)

(c) As in the previous exercise on Werner states, choose again your favorite Π_1, Π_2, calculate $\rho_z|_{\Pi_i}$ and the corresponding probabilities p_i and use these results to calculate $S(\rho_{z\mathbf{X}}|\{\Pi_i\})$. (1P)

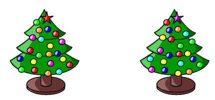
(d) Explain why $S(\rho_{z\mathbf{X}}|\{\Pi_i\})$ does not depend on your particular choice of LPM and conclude that $S(\rho_{z\mathbf{X}}|\{\Pi_i\}) = \inf_{\{\Pi_i\}} S(\rho_{z\mathbf{X}}|\{\Pi_i\})$. (1P)

(e) Use your previous results from above and Eq. (**??**) to show that

$$D_{\rho_z}(\mathbf{X}:\mathbf{Y}) = \frac{1+3z}{4}\ln(1+3z) - \frac{1+z}{2}\ln(1+z) + \frac{1-z}{4}\ln(1-z)$$

and plot $D_{\rho_z}(\mathbf{X}:\mathbf{Y})$ for $0 \leq z \leq 1$. (1P)

Frohes Fest und einen guten Start ins Neue Jahr!



$\mathcal{H}_X \quad \otimes \quad \mathcal{H}_Y$

To be handed in on Monday, January 08, at the beginning of the tutorial. Because of the holidays we will also accept solutions until Wednesday, January 10.