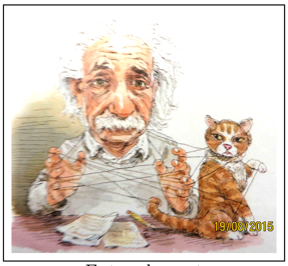

Entanglement
[by Gwydion M. Williams]

Exercise 6.1: Determinant of a tensor product (2P)

Let $\mathbf{A}$ and $\mathbf{B}$ two finite-dimensional square matrices of dimensions $n \times n$ and $m \times m$. Express $\det(\mathbf{A} \otimes \mathbf{B})$ in terms of $\det(\mathbf{A})$ and $\det(\mathbf{B})$ and prove your result in such a way that it even holds for non-diagonalizable matrices.

Exercise 6.2: The Pauli basis (6P)

Let us consider standard Pauli matrices including the identity

$$\sigma_0 = \mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and let us define an N-qubit Pauli operator by the tensor product

$$\sigma_{\mathbf{j}} = \sigma_{j_1} \otimes \sigma_{j_2} \otimes \cdots \otimes \sigma_{j_N} = \bigotimes_{i=1}^{N} \sigma_{j_i}.$$

where $\mathbf{j} = (j_1, j_2, \ldots, j_N)$ is a N-component multiindex with $j_1, \ldots, j_n \in \{0, 1, 2, 3\}$.

(a) Show that $\mathrm{Tr}[\sigma_{\mathbf{j}} \sigma_{\mathbf{k}}] = 2^N \delta_{\mathbf{j},\mathbf{k}}$. (2P)

(b) Let $\mathbf{A}$ be an operator acting on the Hilbert space $\mathbb{C}^{2^N}$. Show that this operator can always be represented by $\mathbf{A} = \sum_{\mathbf{j}} A_{\mathbf{j}} \sigma_{\mathbf{j}}$ and calculate the coefficients $A_{\mathbf{j}}$. (2P)

(c) How are Hermitean operators ($\mathbf{A} = \mathbf{A}^\dagger$) reflected in the components $A_{\mathbf{j}}$? (1P)

(d) Express the scalar product $\langle \vec{A} | \vec{B} \rangle = \sum_{\mathbf{j}} A_{\mathbf{j}}^* B_{\mathbf{j}}$ in terms of $\mathbf{A}$ and $\mathbf{B}$. (1P)

Exercise 6.3: Entanglement of a pure 2-qubit state (4P)

Consider the 2-qubit state $\rho = |\psi\rangle\langle\psi|$ with

$$|\psi\rangle = \frac{1}{5}|00\rangle + \frac{2i\sqrt{2}}{5}|01\rangle - \frac{4}{15}|10\rangle - \frac{8i\sqrt{2}}{15}|11\rangle.$$

(a) Use *Mathematica*® or similar tools to compute the 4×4 matrix ρ in the qubit basis. Verify that ρ is pure. (2P)

(b) Compute the reduced density matrices $\rho^{(L)}$ und $\rho^{(R)}$ for the left and the right qubit by carrying out the partial traces (1P)

$$\rho^{(L)}_{jk} = \sum_{m=0}^{1} \rho_{jm,km} \,, \qquad \rho^{(R)}_{mn} = \sum_{j=0}^{1} \rho_{jm,jn} .$$

(c) A pure state is called *entangled* if the reduced density matrices are mixed. Find out whether ρ is entangled or not. (1P)

(Σ = 12P)

To be handed in on Monday, December 04, at the beginning of the tutorial.