Quantum Information Theory

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The Moyal Medal

Exercise 5.1: Moyal $\star$-product and the Harmonic Oscillator (12P)

In standard quantum mechanics, the ground state $|0\rangle$ and the first excited eigenstate $|1\rangle$ of the harmonic oscillator $\mathbf{H} = \frac{\mathbf{P}^2}{2m} + \frac{m\omega^2\mathbf{Q}^2}{2}$ can be represented by the wave functions

$$\psi_0(q) = \langle q|0\rangle = \frac{1}{\pi^{1/4}a^{1/2}} e^{-\frac{q^2}{2a^2}}\,, \qquad \psi_1(q) = \langle q|1\rangle = \frac{\sqrt{2}}{\pi^{1/4}a^{3/2}}\, q\, e^{-\frac{q^2}{2a^2}}\,,$$

where $a = \sqrt{\frac{\hbar}{m\omega}}$ denotes the characteristic length scale of the oscillator.

(a) Compute the Moyal-Wigner functions $\rho_0(p,q)$ and $\rho_1(q,p)$ by applying the inverse Weyl transform to the pure density matrices $\hat\rho_0 = |0\rangle\langle 0|$ and $\hat\rho_1 = |1\rangle\langle 1|$. (2P)

(b) Check the normalization of the Wigner function $\rho_0(q,p)$. (1P)

(c) Determine the trajectory $\big(q(t),p(t)\big)$ of the *classical* harmonic oscillator for the initial condition $\big(q(0),p(0)\big)$ at $t=0$ by solving the Hamilton equations of motion. Invert the solution in order to express $\big(q(0),p(0)\big)$ in terms of $\big(q(t),p(t)\big)$. (2P)

(d) Since the Moyal-Wigner formalism takes place in phase space, it is near at hand that the time-dependent Wigner function $\rho(q,p,t)$ simply moves like classical particles according to the classical Hamiltonian flow. This means that we can make the ansatz

$$\rho\big(q(t),p(t),\,t\big) = \rho\big(q(0),p(0),\,0\big) \qquad \forall t\,,$$

where $q(t),p(t)$ is a classical solution. Use this ansatz and (c) to prove that (1P)

$$\rho(q,p,t) \;=\; \rho\Big(q\cos\omega t - \frac{p}{m\omega}\sin\omega t\,,\;\; p\cos\omega t + q\,m\omega\sin\omega t\,,\;\; 0\Big).$$

(e) Argue why the star commutator $[H,\rho]_\star$ for ρ computed in (d) is of the form (2P)

$$[H,\rho]_\star = i\hbar\Big((\partial_q H)(\partial_p\rho) - (\partial_q\rho)(\partial_p H)\Big) = i\hbar\{H,\rho\}_{\text{Poisson}}\,.$$

(f) Let us define the abbreviations (1P)

$$A = A(q,p,t) = \frac{\partial\rho(\tilde q,\tilde p,0)}{\partial\tilde q}\bigg|_{\tilde q = q\cos\omega t - \frac{p}{m\omega}\sin\omega t,\; \tilde p = p\cos\omega t + q\,m\omega\sin\omega t}$$

$$B = B(q,p,t) = \frac{\partial\rho(\tilde q,\tilde p,0)}{\partial\tilde p}\bigg|_{\tilde q = q\cos\omega t - \frac{p}{m\omega}\sin\omega t,\; \tilde p = p\cos\omega t + q\,m\omega\sin\omega t}$$

Compute the three partial derivatives $\partial_q\rho(q,p,t)$, $\partial_p\rho(q,p,t)$, and $\partial_t\rho(q,p,t)$.

(g) Use (d)-(f) to prove that for the harmonic oscillator the equation of motion $i\hbar\partial_t\rho = [H,\rho]_\star$ is satisfied, justifying *a posteriori* the ansatz made in (d). (1P)

(h) Show that the ground state function $\rho_0(q,p)$ computed in (a) is invariant along classical trajectories. (1P)

(i) Explain why all eigenstates of $\mathbf{H}$ are represented by constant density matrices $\hat{\rho}$ and constant Moyal-Wigner functions. (1P)

(j) Explain qualitatively the time dependence of a Moyal-Wigner function starting with a 'spatially shifted ground state' of the form

$$\rho_0(q,p) = 2\exp\Big(-\frac{(q-b)^2}{a^2} - \frac{a^2p^2}{\hbar^2}\Big)\,.$$

How does it look like as time evolves? (1P)

(Σ = 12P)

To be handed in on Monday, November 27, at the beginning of the tutorial.