Quantum Information Theory

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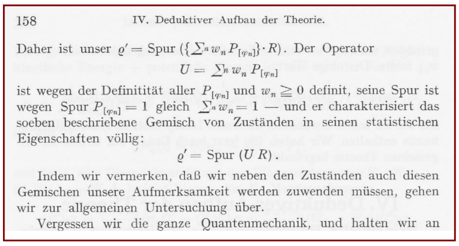
158 IV. Deduktiver Aufbau der Theorie.

Daher ist unser $\varrho' = \text{Spur}\,(\{\sum^n w_n P_{[\varphi n]}\} \cdot R)$. Der Operator

$$U = \sum^n w_n P_{[\varphi n]}$$

ist wegen der Definitität aller $P_{[\varphi n]}$ und $w_n \geqq 0$ definit, seine Spur ist wegen Spur $P_{[\varphi n]} = 1$ gleich $\sum^n w_n = 1$ — und er charakterisiert das soeben beschriebene Gemisch von Zuständen in seinen statistischen Eigenschaften völlig:

$$\varrho' = \text{Spur}\,(U\,R)\,.$$

Indem wir vermerken, daß wir neben den Zuständen auch diesen Gemischen unsere Aufmerksamkeit werden zuwenden müssen, gehen wir zur allgemeinen Untersuchung über.

Vergessen wir die ganze Quantenmechanik, und halten wir an

John von Neumann – Inventor of the density matrix

Exercise 4.1: Density matrix of a single qubit (4P)

For a given ensemble of qubits the observables

$$\mathbf{A} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad \mathbf{C} = \begin{pmatrix} 0 & 2i \\ -2i & 0 \end{pmatrix}$$

have been measured many times, giving the expectation values

$$\langle \mathbf{A} \rangle = 2\,, \quad \langle \mathbf{B} \rangle = 1/2\,, \quad \langle \mathbf{C} \rangle = 0\,.$$

(a) Determine the density matrix of the ensemble.

(b) Is the ensemble pure or mixed?

(c) If we measure σ^z, what is the probability to get $+1$?

(d) Determine the expectation values $\langle \sigma^x \rangle, \langle \sigma^y \rangle, \langle \sigma^z \rangle$.

Exercise 4.2: Criterion for the equivalence of ensembles (8P)

Consider a statistical ensembles $\{|\psi_i\rangle, p_i\}$ with the associated density matrix

$$\rho \;=\; \sum_{i=1}^{N} p_i |\psi_i\rangle\langle\psi_i|.$$

Here N is arbitrary (smaller or larger than the dimension of the Hilbert space) and the $|\psi_i\rangle$ are pairwise different and normalized but not necessarily orthogonal. Moreover, let

$$\rho = \sum_{k=1}^{K} \lambda_k |k\rangle\langle k|$$

be the spectral decomposition of ρ with orthonormal eigenvectors $|k\rangle$ and eigenvalues λ_k. The aim of this exercise is to work out an equivalence theorem for different ensembles with the same density matrix.

(a) Show that every vector $|\psi_i\rangle$ can be written as a linear combination of the eigenvectors $|k\rangle$, that is, prove that $|\psi_i\rangle \in \text{span}\big(\{|k\rangle\}\big)$. (2P)

(b) Use (a) to prove that every $|\psi_i\rangle$ can be written as

$$\sqrt{p_i}|\psi_i\rangle = \sum_{k=1}^{K} c_{ik}\sqrt{\lambda_k}|k\rangle$$

with coefficients c_{ik} obeying $\sum_{i=1}^{N} c_{ik}c_{il}^* = \delta_{k,l}$ for all $k, l = 1, \ldots, K$. (2P)

(c) Because of (a) we know that $N \geq K$. For $N = K$ the result obtained in (b) would imply that the coefficient matrix c_{ik} is unitary. Show that for $N > K$ the rectangular matrix c_{ik} can be extended to a quadratic unitary matrix in such a way that the relation proven in (b) still holds. (1P)

(d) So far we have proven that if $\sum_{i=1}^{N} p_i|\psi_i\rangle\langle\psi_i| = \sum_{k=1}^{K} \lambda_k|k\rangle\langle k|$, then there exists a unitary $N \times N$-matrix c_{ik} such that $\sqrt{p_i}|\psi_i\rangle = \sum_{k=1}^{N} c_{ik}\sqrt{\lambda_k}|k\rangle$. Prove the statement in opposite direction. (1P)

(e) Use (a)-(d) to show that $p_i = \sum_{k=1}^{N} T_{ik}\lambda_k$ (or short: $\vec{p} = \mathbf{T}\vec{\lambda}$), where $\mathbf{T}$ is a so-called *doubly-stochastic matrix*, that is, all matrix entries are real and non-negative and the sums over rows and columns give 1, i.e. $\sum_{i=1}^{N} T_{ik} = \sum_{k=1}^{N} T_{ik} = 1$. (1P)

(f) Use e.g. *Mathematica*® to work out the following explicit example with $N = K = 2$

$$p_1 = p_2 = \frac{1}{2}, \qquad |\psi_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}, \quad |\psi_2\rangle = \begin{pmatrix}0\\1\end{pmatrix}$$

and determine $\lambda_{1,2}$, $|\phi_{1,2}\rangle$, c_{ij}, and T_{ij}. Check the properties derived in (e). (1P)

(Σ = 12P)

To be handed in on Monday, November 20, at the beginning of the tutorial.